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Connections Among Quantum Logics. Part 2. Quantum Event Logics

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This paper gives a brief introduction to the major areas of work in quantum event logics: manuals (Foulis and Randall) and semi-Boolean algebras (Abbott). The two theories are compared, and the connection between quantum event logics and quantum propositional logics is made explicit. In addition, the work on manuals provides us with many examples of results stated in Part I.

1. INTRODUCTION

The study of quantum logic traces to the investigations of Birkhoff and von Neumann (1936), who suggested that the (closed) subspaces of a (separable) Hilbert space may be interpreted as representing the propositions pertaining to a physical system. In the succeeding 50 years, much work has been done in an attempt to clarify this suggestion. The majority of this work falls in the general category of *quantum propositional logics*, and was studied in our paper: Connections Among Quantum Logics; Part I: Quantum Propositional Logics (Lock and Hardegree, 1985).

Some of the most recent and (we think) most exciting research being done today on quantum logic, however, falls in the category of quantum event logics, and it is to that work that we address this paper. The work in quantum event logics, in a way, goes beyond that in quantum propositional logics, in the sense that a quantum propositional logic may be obtained by considered equivalence classes of elements in a quantum event logic. The importance of studying the underlying level of quantum event logics is best illustrated by the probability theory inherent on all these quantum logics. Indeed, many of the subtleties of conditional probability in quantum theory are obscured on quantum propositional logics: As Foulis and Randall have

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shown (Randall and Foulis, 1979), two equivalent events in a quantum event logic may produce different conditional probabilities.

In this paper, we give brief introductions to the study of manuals and the study of semi-Boolean algebras (the two main areas of research in quantum event logics), examine the relations between the two, and illustrate the connection of this approach with that of quantum propositional logics.

2. EMPIRICAL LOGIC: THE STUDY OF MANUALS

We briefly describe here the foundations of the work of Foulis and Randall on manuals. More detailed descriptions and definitions may be found in Foulis and Randall (1979) or Lock and Lock (1984).

Let $\mathscr{A}$ denote a nonempty set of nonempty sets. The elements of the set $\mathscr{A}$ are called *operations*, subsets of operations are called *events*, and elements of operations are called *outcomes*. The set of all events in $\mathscr{A}$ is denoted $\mathscr{E}(\mathscr{A})$ and the set of all outcomes in $\mathscr{A}$ will be denoted by X. Two events, A and B, are called *orthogonal*, denoted $A \perp B$, if they are disjoint and their union is an event. In addition, two events A, B are called *operational complements*, denoted $A \operatorname{oc} B$, if A and B are disjoint and $A \cup B$ is an operation. Two events A, B are called *operationally perspective*, denoted $A \operatorname{op} B$, if there is an event C in $\mathscr{A}$ such that $A \operatorname{oc} C$ and $B \operatorname{oc} C$. The requirement necessary for $\mathscr{A}$ to be a manual is exactly that needed for the relation op to be an equivalence relation on the set of all events:

Definition. A nonempty set of nonempty sets $\mathscr{A}$ is called a *manual* if $A, B, C \in \mathscr{E}(\mathscr{A})$ with $A \operatorname{op} B$ and $B \operatorname{oc} C$ implies $A \operatorname{oc} C$.

A manual is, then, a collection of sample spaces with (perhaps) some overlap of the outcome sets. It is, of course, precisely this overlap that

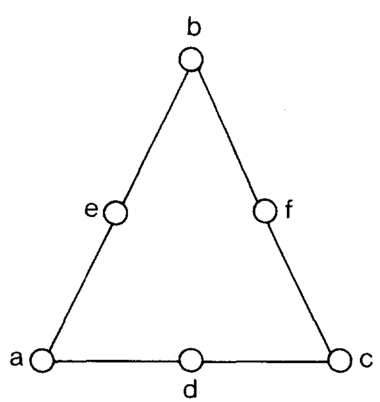

Fig. 1. The Wright triangle.

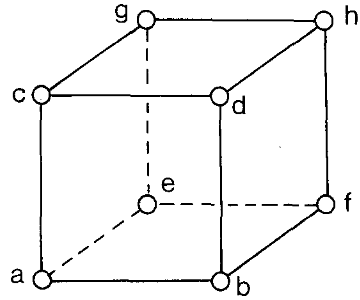

Fig. 2. The Frazer cube.

provides the richness of the theory. An event, in a manual, may have many different (but "equivalent") complements. The following two examples of manuals will be useful to us later. One can check that the manual condition is satisfied in each case.

Examples. 1. The *Wright triangle* manual is illustrated in Figure 1. Operations are connected by a straight line. Hence, this manual contains three operations, each of which consists of three outcomes.

2. The *Frazer cube* manual is illustrated in Figure 2. In this case, the operations are the faces of the cube. We see that there are six operations, each consisting of four outcomes.

3. OPERATIONAL LOGIC: THE LOGIC OF A MANUAL

The logic associated with a manual $\mathscr{A}$, called its *operational logic* and denoted $\pi(\mathscr{A})$, is the set of all events modulo the equivalence relation op. We denote the equivalence class containing event A by $p(A)$, i.e., $p(A) = \{B \in \mathscr{E}(\mathscr{A})\colon B \text{ op } A\}$. The elements $p(A)$ are called *propositions.* We define the following structure on $\pi(\mathscr{A})$: 0 is defined to be $p(\phi)$ and 1 is defined to be $p(E)$ for any operation $E \in \mathscr{A}$. For any proposition $p(A)$, we define $p(A)'$ to be $p(C)$, where C is any event such that $A \text{ oc } C$. We say two propositions are orthogonal, $p(A) \perp p(B)$, if and only if $A \perp B$, and if $p(A) \perp p(B)$ we define $p(A) \oplus p(B) = p(A \cup B)$. It is easily seen that these definitions are all well defined.

Theorem. L is an associative ortho-algebra as defined in Part 1 (Lock and Hardegree, 1985) if and only if L is the logic associated with some manual.

This theorem makes precise the connection between manuals (or quantum event logics) and operational logics (or quantum propositional logics). To obtain the associated manual in the proof of the theorem, we construct the "manual of finite partitions of unity in L" as follows: It is the set of all finite, nonempty, jointly orthogonal subsets E of L such that $0 \notin E$ and $\sum E = 1$. It can be checked that this construction gives a manual whose operational logic is L. The remainder of the proof of the theorem is easily verified.

In Part 1 of this paper, we indicated that we would postpone many of the examples and counterexamples mentioned there until Part 2. They are included below. The proof of each of these facts is left to the reader.

Example. Let $\mathscr{A}_1$ denote the Wright triangle as shown in Figure 1 earlier. Then $\pi(\mathscr{A}_1)$ is its associated operational logic.

1. In an associative ortho-algebra, pairwise orthogonal does not imply jointly orthogonal. In $\pi(\mathscr{A}_1)$, the set $\{p(a), p(b), p(c)\}$ is pairwise orthogonal but not jointly orthogonal.
2. In an associative ortho-algebra, pairwise compatible does not imply jointly compatible. The example given in point 1 above illustrates this as well.
3. In an associative ortho-algebra, the unique Mackey decomposition property does not imply orthocoherence. It can be shown that $\pi(\mathscr{A}_1)$ satisfies the UMD property. However, example 1 above indicates that $\pi(\mathscr{A}_1)$ is not orthocoherent.
4. In an associative ortho-algebra, $a \oplus b$ is a local supremum for a and b (that is, the supremum in a given block containing both a and b), but it need not be a global supremum for a and b. Notice in $\pi(\mathscr{A}_1)$ that $p(a) \leq p(\{a, f\})$ and $p(b) \leq p(\{a, f\})$, but $p(a) \oplus p(b) \nleq p(\{a, f\})$.

Example. Let $\mathscr{A}_2$ denote the Frazer cube as shown in Figure 2 earlier. Then $\pi(\mathscr{A}_2)$ is its associated operational logic.

1. In an associative ortho-algebra, two elements may be compatible but not uniquely compatible. Notice that $p(\{c, d\})$ and $p(\{b, d\})$ are compatible in $\pi(\mathscr{A}_2)$ since $\{p(b), p(c), p(d)\}$ is a Mackey decomposition. However, $p(\{c, d\}) = p(\{e, f\})$ and $p(\{b, d\}) = p(\{e, g\})$, and we see that $\{p(e), p(f), p(g)\}$ gives a second (different) Mackey decomposition.
2. There exists a Boolean atlas which is not a Boolean manifold. Let $\mathscr{B}$ denote the set of blocks on $\pi(\mathscr{A}_2)$. Then $\mathscr{B}$ is a Boolean atlas. Let $B_1 = \{p(D): D \subseteq \{c, d, e, f\}\}$ and $B_2 = \{p(D): D \subseteq \{a, b, g, h\}\}$. It can be shown that $B_1, B_2 \in \mathscr{B}$. Note that $p(\{c, e\}) = p(\{b, h\}) \in B_1 \cap B_2$ and $p(\{c, d\}) = p(\{g, h\}) \in B_1 \cap B_2$. However, $M_1(p(\{c, e\}), p(\{c, d\})) = p(c) \neq p(h) = M_2(p(\{b, h\}), p(\{g, h\}))$ and $J_1(p(\{c, e\}), p(\{c, d\})) = p(\{c, d, e\}) \neq p(\{b, g, h\}) = J_2(p(\{b, h\}), p(\{g, h\})$.

4. SEMI-BOOLEAN ALGEBRAS

Semi-Boolean algebras, it turns out, may be viewed as a natural generalization of manuals. In addition, if we construct the logic associated with such a generalization in a similar manner, we again obtain an associative ortho-algebra.

Again, we give only a brief introduction here. These definitions are borrowed and adapted from Abbott (1967). A *meet semi-Boolean algebra* is a meet semilattice in which every principal ideal is a Boolean algebra. (Abbott has shown that meet semi-Boolean algebras are equationally definable as subtraction algebras.) We are interested in meet semi-Boolean algebras which satisfy the following additional property.

Definition. A meet semi-Boolean algebra S is said to have the *maximal element property* if there is a subset M of S satisfying the following conditions: (i) For all $m \in M$ and all $s \in S$, if $m \leq s$ then $m = s$. (ii) For all $s \in S$, there is an $m \in M$ with $s \leq m$. The set M is called the set of *maximal elements.*

A *semi-Boolean algebra* is defined to be a meet semi-Boolean algebra with the maximal element property. We define a *block* of a semi-Boolean algebra to be a maximal principal ideal (or, equivalently, the ideal generated by a maximal element). Notice that every semi-Boolean algebra is covered by its blocks, and hence may be viewed as a "pasted" family of Boolean algebras (although in quite a different sense from our Boolean atlases of quantum propositional logics). Any two blocks are pasted along a common principal ideal (and nothing else), which minimally includes the zero element.

In order to continue our efforts to compare manuals and semi-Boolean algebras, we wish to define the equivalent notion of the manual condition on semi-Boolean algebras. To that end, we define the following partial two-place operation:

Definition. The *relative complement* of a with respect to m, denoted m/a, is defined if and only if m is a maximal element and $a \leq m$. In this case, m/a is the complement of a in the Boolean algebra generated by m.

Notice that a given element $a \in S$ may have several different relative complements. We say that two elements $a, b \in S$ are *perspective* if they share a relative complement, and we say they are *strongly perspective* if they have precisely the same relative complements. A semi-Boolean algebra S is called *perspectively coherent* if every perspective pair of elements is strongly perspective.

Theorem. 1. The set of events of a manual ordered by inclusion is a perspectively coherent semi-Boolean algebra.

2. A perspectively coherent semi-Boolean algebra in which every block is a complete atomic Boolean algebra naturally defines a manual.

Both of these facts follow directly from the definitions.

We have shown that the operational logic associated with a manual is an associative ortho-algebra. We may construct the set of equivalence classes associated with a perspectively coherent semi-Boolean algebra in a completely analogous way. It can be shown that the "logics" obtained in this way are, also, associative ortho-algebras. In fact, no new logics are obtained by considering the set of equivalence classes under perspectivity of elements of a perspectively coherent semi-Boolean algebra. Hence, we may think of associative ortho-algebras as the quotient structure associated with either a manual or a perspectively coherent semi-Boolean algebra. We emphasize here that there are many manuals, and, more generally, many perspectively coherent semi-Boolean algebras, associated with each associative ortho-algebra.

5. SUMMARY

While the bulk of the work done thus far on quantum logics has dealt with quantum propositional logics, the current work on quantum event logics offers a new richness to the theory and we expect that most of the work done in the future will include this rapidly expanding area of quantum logics. Thus, it is important to understand quantum event logics and the role they play. We have compared the two major areas of work in quantum event logics, and shown how this work fits in with the earlier work on quantum propositional logics.

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